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Fluctuations Around the Boltzmann Equation

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For a system of hard spheres we prove the convergence of the second moment of the fluctuation field in the low-density limit. This extends a previous result by van Beijeren, Lanford, Lebowitz and Spohn$^{(1)}$ to nonequilibrium states.

KEY WORDS: Hard sphere gas; Grad limit; fluctuation theory.

1. INTRODUCTION TO THE PROBLEM OF FLUCTUATIONS

We consider a system of hard spheres of diameter ϵ and unit mass inside a bounded region $\Lambda \subset \mathbb{R}^3$. The hard spheres collide elastically amongst themselves and are specularly reflected at the boundary of Λ. In essence this prescription defines the dynamics of hard spheres. We denote the corresponding flow by T_t^ϵ acting on the grand canonical phase space $\Gamma = \cup_{n \geqslant 0}(\Lambda \times \mathbb{R}^3)^n$. If $x \in \Gamma$ stands for the initial positions and momenta of the particles, then $T_t^\epsilon x$ are the positions and momenta of the particles at the time t. We assume that the initial state of the system is given by a probability measure μ^ϵ on Γ which is absolutely continuous with respect to the Lebesgue measure. (In Γ there are configurations for which spheres overlap. Initially, and therefore at any other time, probability zero is assigned to these configurations. There are also other configurations leading in the course of time to grazing and triple collisions. For these T_t^ϵ remains undefined. As to be discussed in the following section they form a set of measure zero with respect to μ^ϵ.)

We want to understand the macroscopic behavior of the hard sphere gas at low density. In this regime it is natural to study the number $n^\epsilon(\Delta, t)$ of particles in Δ at time t, where $\Delta \subset \Lambda \times \mathbb{R}^3$ is some region of the

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one-particle phase space. For given initial conditions $n^\epsilon(\Delta, t)$ is some well-defined integer. However, since we assumed a distribution of the initial conditions according to the probability measure μ^ϵ, $n^\epsilon(\Delta, t)$ has to be considered a random variable on (Γ, μ^ϵ). Note that the randomness enters only through the initial conditions. The dynamics is deterministic.

Let $x_j = (q_j, p_j)$ stand for the position and momentum of the jth particle and let χ_Δ denote the indicator function of the set Δ.

Then, with $n^\epsilon(\Delta, 0) = n^\epsilon(\Delta)$,

$$n^\epsilon(\Delta)(x_1, \ldots, x_n) = \sum_{j=1}^{n} \chi_\Delta(x_j) \tag{1.1}$$

and

$$\begin{aligned} n^\epsilon(\Delta, t)(x_1, \ldots, x_n) &= \left[n^\epsilon(\Delta) \circ T_t^\epsilon \right](x_1, \ldots, x_n) \\ &= n^\epsilon(\Delta)\left[T_t^\epsilon(x_1, \ldots, x_n) \right] \end{aligned} \tag{1.2}$$

Both physically as well as mathematically it is convenient to consider a somewhat wider class of random variables. Let $f: \Lambda \times \mathbb{R}^3 \to \mathbb{R}$ be a bounded and measurable function on the one-particle space. Then we define

$$X^\epsilon(f)(x_1, \ldots, x_n) = \sum_{j=1}^{n} f(x_j) \tag{1.3}$$

on (Γ, μ^ϵ) and

$$X^\epsilon(f, t) = X^\epsilon(f) \circ T_t^\epsilon \tag{1.4}$$

The collection of random variables $\{X^\epsilon(f, t) \mid f \in L^\infty(\Lambda \times \mathbb{R}^3), t \in \mathbb{R}\}$ is a generalized *random field over* $\Lambda \times \mathbb{R}^3 \times \mathbb{R}$.

Often, a fixed countable partition $\{\Delta_i \mid i \in \mathbb{N}\}$ of the one-particle space is introduced by the argument that physical measurements have only a finite resolution. This construction is physically rather artificial and in fact unnecessary. What one really wants to study is the number of particles $n^\epsilon(q, p, t)$ at the point $(q, p) \in \Lambda \times \mathbb{R}^3$ at time t considered as random variables. Then $n^\epsilon(q, p, t)$ is a random field over $\Lambda \times \mathbb{R}^3 \times \mathbb{R}$. Since $n^\epsilon(q, p, t)$ is a distribution rather than a function on Γ we follow the common practice to integrate $n^\epsilon(q, p, t)$ over an arbitrary test function f in order to obtain the well-defined random variables $X^\epsilon(f, t) = \int dq\, dp\, n^\epsilon(q, p, t) f \cdot (q, p)$.

We want to investigate the structure of the random field $X^\epsilon(f, t)$ at low (volume) density. As discussed before[2,3] the proper scaling is the Grad limit

$$\text{diameter} = \epsilon$$

$$\text{particle density} \sim \epsilon^{-2}$$

with $\epsilon \to 0$, which implies a constant mean free path and that the fraction of volume occupied by spheres is proportional to ϵ. The increase of the particle density is a condition on the initial measure μ^ϵ. We denote the average with respect to μ^ϵ by $\langle \cdot \rangle_\epsilon$. Then Lanford[3-5] has proved, under suitable assumptions on μ^ϵ (cf. Section 2), the following theorem: If at $t = 0$

$$\lim_{\epsilon\to 0} \epsilon^2 \langle X^\epsilon(f)\rangle_\epsilon = \int dq\, dp\, r(q, p) f(q, p) \tag{1.5}$$

and

$$\lim_{\epsilon\to 0} \epsilon^4 \left\{ \left\langle X^\epsilon(f)^2 \right\rangle_\epsilon - \left\langle X^\epsilon(f)\right\rangle_\epsilon^2 \right\} = 0 \tag{1.6}$$

then also for $|t| \leqslant t_0$

$$\lim_{\epsilon\to 0} \epsilon^2 \langle X^\epsilon(f, t)\rangle_\epsilon = \int dq\, dp\, r(q, p, t) f(q, p) \tag{1.7}$$

and

$$\lim_{\epsilon\to 0} \epsilon^4 \left\{ \left\langle X^\epsilon(f, t)^2 \right\rangle_\epsilon - \left\langle X^\epsilon(f, t)\right\rangle_\epsilon^2 \right\} = 0 \tag{1.8}$$

$r(q, p, t)$ is the solution of the Boltzmann equation with initial conditions $r(q, p)$, which are defined through (1.5). For negative times the sign of the collision operator in the Boltzmann equation has to be reversed. t_0 depends on the sequence of initial states. The restriction $|t| \leqslant t_0$ is believed to be of a technical nature.

Lanford's result means that the distribution of $\epsilon^2 X^\epsilon(f, t)$ converges to a δ function concentrated at $\int dq\, dp\, r(q, p, t) f(q, p)$. For low density the random field $\epsilon^2 X^\epsilon(f, t)$ becomes deterministic and its evolution is governed by a nonlinear field equation on the one-particle phase space.

Given Lanford's result the formulation of the fluctuation problem is obvious. One defines the fluctuation field

$$\xi^\epsilon(f, t) = \epsilon\left[X^\epsilon(f, t) - \langle X^\epsilon(f, t)\rangle_\epsilon \right] \tag{1.9}$$

and one would like to know whether

$$\lim_{\epsilon\to 0} \xi^\epsilon(f, t) = \xi(f, t) \tag{1.10}$$

exists.

One part of the problem is to guess the structure of the limiting field. Quite generally, it is assumed that the limiting field is Gaussian. Then the problem is reduced to finding the covariance of $\xi(f, t)$. If μ^ϵ is a sequence of grand canonical equilibrium states with fugacity $z_\epsilon = \epsilon^{-2} z$ and inverse temperature β, then the correct covariance may be guessed through the knowledge of the stationary state. (This is often referred to as the fluctuation-dissipation theorem.) If μ^ϵ is a sequence of nonequilibrium states, then the problem is more subtle. A careful formal discussion may be found in a recent preprint by Cohen and Ernst[6] with references to earlier work.

To uncover the structure of the limiting field is not a purely academic problem. Light-scattering measurements allows us to measure density fluctuations with great precision. In this connection the correct prediction for the fluctuations in a low-density gas in a steady heat-transporting state has caused some discussions recently.[7–11]

The fluctuation problem is well known from other dynamical systems. Braun and Hepp[12] studied an interacting particle system in the mean field limit. The properly scaled number of particles in some region of the one-particle phase space is governed by the Vlasov equation. They prove the existence of a limiting fluctuation field and show that this field is Gaussian. The fluctuations present in the initial state evolve deterministicly according to the linearized Vlasov equation. The mean field limit of quantum mechanical models has been studied by Hepp and Lieb.[13] The limiting fluctuation field is a free Bose field in vacuum.

For stochastic models there are many examples for the kind of fluctuation problem considered here. To mention only some of them: the Ω expansion of van Kampen[14] and its rigorous treatment by Kurtz,[15] stochastic Ising models as studied by Holley and Strook,[16] and stochastic models for chemical reactions as discussed by Nicolis and Prigogine[17] and rigorized by Arnold and Kotelenez.[18]

Let us point out that equilibrium fluctuation theory also follows the above scheme. Away from phase transitions, the distribution of intensive observables, as the energy per volume or the number of particles per volume, tends to a δ function at a point determined by the free energy per unit volume. Their fluctuations are Gaussian and are determined by suitable second derivatives of the free energy.

For the Boltzmann equation the convergence of the second moment of the fluctuation field for a sequence of grand canonical equilibrium states with fugacity $z_\epsilon = \epsilon^{-2} z$ and inverse temperature β has been proved in Ref. 1. Here we prove the convergence of the second moment of the fluctuation field at unequal times for a sequence of nonequilibrium states. In both cases it can be proved that in the limit $\epsilon \to 0$ the third moment vanishes and that the fourth moment converges to pairings of the limiting second moments as should be the case for a Gaussian random field. The convergence of higher moments has not yet been investigated in detail. Under sufficiently strong assumptions on the initial state Gaussian fluctuations seem to be plausible.

2. HARD SPHERE DYNAMICS: LOW-DENSITY LIMIT

The dynamics of hard spheres is particularly simple, since the collision time is zero. On the other hand there are certain exceptional initial conditions for which the dynamics remains undefined. For example, if three spheres touch each other it would be unwise to find a constructive rule for how to continue the dynamics through such a triple collision. A careful discussion of the dynamics of hard spheres may be found in the thesis of Alexander[19] (cf. also Aizenman[20]). We will establish here just some notation and quote some results which will be needed later on.

We assume that Λ is a bounded region of $\mathbb{R}^3$ with smooth boundary $\partial\Lambda$ with a curvature strictly bounded below and bounded above by ϵ^{-1}. Let $\Gamma_n = (\Lambda \times \mathbb{R}^3)^n$. The allowed phase space is

$$\Gamma_n(\epsilon) = \{x_1, \ldots, x_n \in \Gamma_n \mid |q_i - q_j| \geqslant \epsilon, |q_i - q| \geqslant \epsilon/2,$$
$$q \in \partial\Lambda, i \neq j = 1, \ldots, n\}$$

We define a mapping $x \in \partial\Gamma_n(\epsilon)$ to $x' \in \partial\Gamma_n(\epsilon)$ relating the phase point x after the collision to the phase point x' before the collision. Let $x = (x_1, \ldots, x_n) \in \partial\Gamma_n(\epsilon)$. If either $|q_i - q_j| = \epsilon$, $|q_j - q_k| = \epsilon$ or $|q_i - q_j| = \epsilon/2$, $|q_i - q_j| = \epsilon$ for $q \in \partial\Lambda$, $i \neq j \neq k = 1, \ldots, n$, then x' is not defined (triple collisions). Let $\hat{n}(q)$ be the inward normal to $\partial\Lambda$ at q and let $\hat{\omega}$ be a unit vector in $\mathbb{R}^3$, $\hat{\omega} \in S^2$. If either $q_j = q_i + \epsilon\hat{\omega}$ and $\hat{\omega} \cdot (p_i - p_j) = 0$ or $q_i = q + (\epsilon/2)\hat{n}(q)$ and $\hat{n}(q) \cdot p_i = 0$, then x' is not defined (grazing collisions). For a pair collision between particle i and j, x and x' are related by

$$x_i = (q_i, p_i), \qquad x_j = (q_i + \epsilon\hat{\omega}, p_j)$$
$$x_i' = (q_i, p_i'), \qquad x_j' = (q_i + \epsilon\hat{\omega}, p_j')$$

with $\hat{\omega} \cdot (p_i - p_j) < 0$ and

$$\begin{aligned} p_i' &= p_i - [\hat{\omega} \cdot (p_i - p_j)]\hat{\omega} \\ p_j' &= P_j + [\hat{\omega} \cdot (p_i - p_j)]\hat{\omega} \end{aligned} \tag{2.1}$$

If $\hat{\omega} \cdot (p_i - p_j) > 0$, then $x_i = x_i'$, $x_j = x_j'$.

For a specular reflection of particle i at the boundary x and x' are related by

$$x_i = (q_i, p_i), \qquad x_i' = (q_i, p_i')$$

with $\hat{n}(q) \cdot p_i > 0$ and

$$p_i' = p_i - 2[\hat{n}(q) \cdot p_i]\hat{n}(q) \tag{2.2}$$

If $\hat{n}(q) \cdot p_i < 0$, then $x_i = x_i'$.

The flow T_t^ϵ is now constructed in the obvious way. It will turn out to be convenient to define T_t^ϵ as continuous from the future. For $t \geqslant 0$, let $x(-t) = (q_1 - p_1 t, p_1, \ldots, q_n - p_n t, p_n)$ according to the free motion. Let $t_1 \geqslant 0$ be the smallest time such that $x(-t_1) \in \partial\Gamma_n(\epsilon)$. Then $T_{-t}^\epsilon x = x(-t)$ for $0 \leqslant t \leqslant t_1$. If $x'(-t_1)$ is defined, then $x'(-t_1 - t) = q_1(-t_1) - p_1'(-t_1)t$, $p_1'(-t_1), \ldots, q_n'(-t_1) - p_n'(-t_1)t, p_n'(-t_1)$. Let t_2 be the smallest time such that $x'(-t_1 - t_2) \in \partial\Gamma_n(\epsilon)$. Since $x'(-t_1)$ points to the exterior of $\Gamma_n(\epsilon)$, necessarily $t_2 > 0$. Then $T_{-(t_1+t)}^\epsilon x = x'(-t_1 - t)$ for $0 < t \leqslant t_2$, etc. By this construction $T_{-t}^\epsilon x$ is defined until a grazing collision or a triple collision is reached. Let $T > 0$. Then

$$\Gamma_n(\epsilon, T) = \{x \in \Gamma_n(\epsilon) \mid T^\epsilon_{-t}x \text{ is defined for } 0 \leqslant t \leqslant T\}$$

Alexander[19] proves that $\Gamma_n(\epsilon)\backslash\Gamma_n(\epsilon, T)$ has Lebesgue measure zero.

We impose now conditions on the initial measure μ^ϵ. We assume that μ^ϵ is absolutely continuous with respect to the Lebesgue measure, i.e.,

$$\mu^\epsilon(dx_1, \ldots, dx_n) = f^\epsilon_n(x_1, \ldots, x_n)\frac{1}{n!}\, dx_1 \cdots dx_n \tag{2.3}$$

Then $f^\epsilon_n \circ T^\epsilon_{-t} = f^\epsilon_n(t)$, defined on $\Gamma_n(\epsilon, t)$, are the densities of the time-evolved measure $\mu^\epsilon \circ T^\epsilon_{-t}$. The correlation functions corresponding to this measure are defined by

$$\rho^\epsilon_n(x_1, \ldots, x_n, t) = \sum_{m=0}^{\infty} \frac{1}{m!} \int dx_{n+1} \cdots dx_{n+m}\, f^\epsilon_{n+m}(x_1, \ldots, x_{n+m}, t) \tag{2.4}$$

on $\Gamma_n(\epsilon, t)$. For the Grad limit it is convenient to consider the rescaled correlation functions defined by

$$r^\epsilon_n(t) = \epsilon^{2n}\rho^\epsilon_n(t) \tag{2.5}$$

We will use the vector notation $r^\epsilon(t) = (r^\epsilon_0(t), r^\epsilon_1(t), \ldots)$ for the sequence of correlation functions and we set $r^\epsilon(0) = r^\epsilon$.

Let h_β be the normalized Maxwellian,

$$h_\beta(x_j) = (\beta/2\pi)^{3/2} exp\left(-\tfrac{1}{2}\beta p_j^2\right).$$

Then we assume the following.

(C1) There exist constants M, β, and z such that

$$r^\epsilon_n(x_1, \ldots, x_n) \leqslant M \prod_{j=1}^{n} \{zh_\beta(x_j)\} \tag{2.6}$$

on $\Gamma_n(\epsilon)$ for all $n = 1, 2, \ldots$.

Note that (C1) implies a corresponding bound on $f^\epsilon_n(t)$ and $r^\epsilon_n(t)$, however, with a constant $M(\epsilon) = M\exp(\epsilon^{-2}z)$.

Let $S^\epsilon_n(t)r_n = r_n \circ T^\epsilon_{-t}$ be the group induced by the dynamics of n spheres. Let C^ϵ_{n+1} be the collision operator defined by $(C^\epsilon_{n+1}r^\epsilon)_j = \delta_{nj}C^\epsilon_{n+1}r^\epsilon_{n+1}$ and

$$\begin{aligned}(C^\epsilon_{n+1}r^\epsilon_{n+1})(x_1, \ldots, x_n) = \sum_{j=1}^{n} \int_{\mathbb{R}^3 \times S^2} dp_{n+1}\, d\hat{\omega} \\ \times\, \hat{\omega} \cdot (p_{n+1} - p_j) r^\epsilon_{n+1}(x_1, \ldots, q_j, p_j, \ldots, q_j + \epsilon\hat{\omega}, p_{n+1})\end{aligned} \tag{2.7}$$

Then Lanford proves that on $\Gamma_n(\epsilon, t)$

$$\begin{aligned}r^\epsilon_n(x_1, \ldots, x_n, t) &= \left[S^\epsilon_n(t)r^\epsilon_n\right](x_1, \ldots, x_n) \\ &\quad + \int_0^t ds\, S^\epsilon_n(t-s)\left[C^\epsilon_{n+1}r^\epsilon_{n+1}(s)\right](x_1, \ldots, x_n)\end{aligned} \tag{2.8}$$

Iterating (2.8) yields the perturbation expansion

$$r_n^\epsilon(t) = \sum_{m=0}^{\infty} \int_{0 \leqslant t_m \cdots \leqslant t_1 \leqslant t} dt_1 \cdots dt_m \, S_n^\epsilon(t-t_1) \times C_{n+1}^\epsilon \cdots C_{n+m}^\epsilon S_{n+m}^\epsilon(t_m) r_{n+m}^\epsilon \tag{2.9}$$

(2.9) is valid for all times. In fact (2.9) is a finite sum because of the hard core exclusion. Because of the Maxwellian bound (C1) we may manipulate (2.9) freely as long as $\epsilon > 0$.

We are now in a position to make precise the low-density behavior, $\epsilon \to 0$, of the correlation functions $r_n^\epsilon(t)$. Let $S_n(t)f = f \circ T_{-t}^{(n)}$, where $T_{-t}^{(n)}$ is the flow corresponding to n free particles inside Λ with specular reflection at $\partial\Lambda$. Let

$$\tilde{\Gamma}_n(t) = \left\{ x_1, \ldots, x_n = x \in \Gamma_n \,|\, q_i(s,x) \neq q_j(s,x) \text{ for } 0 \leqslant s \leqslant t \text{ and } i \neq j = 1, \ldots, n, \text{ where } q_j(s,x) = q_j\left(T_{-s}^{(n)}x\right) \right\}$$

Points in $\tilde{\Gamma}_n(t)$ do not lead under the free motion backwards in time to a collision between any pair of particles, regarded as point particles. Let the limiting collision operator be defined by

$$(C_{n+1}r)_j = \delta_{nj} C_{n+1} r_{n+1}$$

and

$$\begin{aligned}(C_{n+1}r_{n+1})(x_1, \ldots, x_n) &= \sum_{j=1}^{n} \int_{\hat{\omega}\cdot(p_{n+1}-p_j) \geqslant 0} dp_{n+1}\, d\hat{\omega}\, \hat{\omega}\cdot(p_{n+1}-p_j) \\ &\quad \times \left[r_{n+1}(x_1, \ldots, q_j, p_j', \ldots, q_j, p_{n+1}') \right. \\ &\qquad \left. - r_{n+1}(x_1, \ldots, q_j, p_j, \ldots, q_j p_{n+1}) \right]\end{aligned} \tag{2.10}$$

To ensure the existence of the Grad limit $\epsilon \to 0$ in addition to (C1) we have to impose the following.

(C2) There exists a function r_n on Γ_n which is continuous and continuous through specular reflection at $\partial\Lambda$ such that for some $s \geqslant 0$

$$\lim_{\epsilon \to 0} r_n^\epsilon = r_n \tag{2.11}$$

uniformly on compact sets of $\tilde{\Gamma}_n(s)$ for all $n = 1, 2, \ldots$.

Theorem (Lanford). Let (C1) and (C2) be satisfied. Then for

$$\begin{gathered} 0 \leqslant t \leqslant t_0 = t_0(z, \beta) = 0.1\sqrt{\beta}/z, \\ \lim_{\epsilon \to 0} r_n^\epsilon(t) = r_n(t) \end{gathered} \tag{2.12}$$

exists uniformly on compact sets of $\tilde{\Gamma}_n(t+s)$. $r_n(t)$ is continuous and satisfies (2.6) with some M', z', β'. In this class of functions $r_n(t)$ is the unique solution of the set of integral equations

$$r_n(t) = S_n(t)r_n + \int_0^t ds\, S_n(t-s)C_{n+1}r_{n+1}(s), \qquad n = 0, 1, \ldots \tag{2.13}$$

3. SECOND MOMENT OF THE FLUCTUATION FIELD

From now on we assume that Λ is a box with periodic boundary conditions ($\equiv$ torus). So points in $\partial\Lambda \times \mathbb{R}^3$ are identified in the obvious way. We should distinguish this notationally as $(\Lambda \times \mathbb{R}^3)_{\text{periodized}}$, but we will not do so.

For the existence of the second moment of the fluctuation field we need stronger assumptions on the correlation functions of the initial state.

(F1) There exist constants M', z', β' such that for all $m, n \geqslant 1$

$$\left|\epsilon^{-2}\left[r^{\epsilon}_{n+m}(x_1, \ldots, x_{n+m}) - r^{\epsilon}_n(x_1, \ldots, x_n)r^{\epsilon}_m(x_{n+1}, \ldots, x_{n+m})\right]\right|$$
$$\leqslant M' \prod_{j=1}^{n+m} \{z' h_{\beta'}(x_j)\} \tag{3.1}$$

Note that there is no hope for (3.1) being satisfied outside $\Gamma_{n+m}(\epsilon)$.

(F2) There exist continuous functions $r: \Lambda \times \mathbb{R}^3 \to R$ and $h: (\Lambda \times \mathbb{R}^3)^2 \to R$ such that for all $m, n \geqslant 1$,

$$\lim_{\epsilon\to 0} \epsilon^{-2}\left[r^{\epsilon}_{n+m}(x_1, \ldots, x_{n+m}) - r^{\epsilon}_n(x_1, \ldots, x_n)r^{\epsilon}_m(x_{n+1}, \ldots, x_{n+m})\right]$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{m} \left\{ h(x_j, x_{n+i}) \prod_{\substack{k=1 \\ k\neq j,\, k\neq n+i}}^{n+m} r(x_k) \right\} \tag{3.2}$$

uniformly on compact sets of $\Gamma_{n+m}(0)$.

We note that (F1) and (F2) imply

$$\lim_{\epsilon\to 0} \langle \xi^{\epsilon}(f)\xi^{\epsilon}(g)\rangle_{\epsilon} = \int dx_1\, dx_2\, h(x_1, x_2) f(x_1) g(x_2)$$
$$+ \int dx_1\, f(x_1) g(x_1) r(x_1) \tag{3.3}$$

So $h(x_1, x_2) + \delta(x_1 - x_2) r(x_1)$ is the kernel of the covariance matrix of the initial field. In particular,

$$\lim_{\epsilon\to 0} \epsilon^4\left[\left\langle X^{\epsilon}(f)^2\right\rangle_{\epsilon} - \left\langle X^{\epsilon}(f)\right\rangle_{\epsilon}^2\right] = 0 \tag{3.4}$$

which implies by (C1) and (C2)

$$r_n(x_1, \ldots, x_n) = \prod_{j=1}^{n} r(x_j) \tag{3.5}$$

in (2.11). For the initial conditions (3.5) the solution of the integral equations (2.13) is given by

$$r_n(x_1, \ldots, x_n, t) = \prod_{j=1}^{n} r(x_j, t)$$

where $r(x_j, t)$ is the solution of the *Boltzmann equation* with initial conditions $r(x_j)$,

$$r(t) = S_1(t)r + \int_0^t ds\, S_1(t-s)C_2 r(s)r(s) \tag{3.6}$$

(C1) and (C2) imply $r(q, p) \leqslant Mzh_\beta(p)$. Then (F1) and (F2) imply $h(x_1, x_2) = h(x_2, x_1)$ and $h(x_1, x_2) \leqslant M'z'^2 h_{\beta'}(x_1)h_{\beta'}(x_2)$.

If μ^ϵ is the sequence of grand canonical equilibrium states with fugacity $z_\epsilon = \epsilon^{-2}z$ and inverse temperature β, then using the Mayer expansion one shows that (F1) and (F2) are satisfied with $h(x_1, x_2) = 0$, $r(q, p) = zh_\beta(p)$, and $z' = ez$, $\beta' = \beta$.

To state our result we need still some further definitions. Let $C_{r(t)}$ be the collision operator linearized at $r(t)$,

$$\begin{aligned}(C_{r(t)}f)(q, p) = \int_{\hat\omega\cdot(p_1 - p)\geqslant 0} & dp_1\, d\hat\omega\, \hat\omega\cdot(p_1 - p) \\ & \times\left[r(q, p_1', t)f(q, p') + r(q, p', t)f(q, p_1')\right. \\ & \left. - r(q, p_1, t)f(q, p) - r(q, p, t)f(q, p_1)\right]\end{aligned} \tag{3.7}$$

Since $r(t)$ is defined only for $0 \leqslant t \leqslant t_0(z, \beta)$, so is $C_{r(t)}$. We define the propagator $U(t,s)$ as the solution operator of the *linearized Boltzmann equation* in its integral form

$$U(t,s)f = S_1(t-s)f + \int_s^t ds'\, S_1(t-s')C_{r(s')}\left[U(s',s)f\right] \tag{3.8}$$

Let C_h be the class of continuous functions on $\Lambda \times \mathbb{R}^3$ bounded by some Maxwellian. Then in C_h the integral equation (3.8) has a unique solution for $0 \leqslant s \leqslant t \leqslant t_0(z, \beta)$. The same is true for the solution $U^*(t,s)$ of the adjoint equation.

We define for $f, g \in C_h$ the *recollision operator*

$$\begin{aligned}\left[R(f, g)\right](q_1, p_1, q_2, p_2) &= \delta(q_1 - q_2) \\ \int_{\hat\omega\cdot(p_2 - p_1)\geqslant 0} & d\hat\omega\, \hat\omega\cdot(p_2 - p_1)\{f(q_2, p_2')g(q_1, p_1') \\ & - f(q_2, p_2)g(q_1, p_1)\}\end{aligned} \tag{3.9}$$

Theorem. Let the correlation functions of the sequence of initial states μ^ϵ satisfy (C1), (C2), (F1), and (F2). Let $f, g \in C_h$. Then for $0 \leqslant s \leqslant t$

$\leqslant t_0 = \min[t_0(z, \beta), t_0(z', \beta')]$

$$\lim_{\epsilon\to 0} \langle \xi^\epsilon(f,t)\xi^\epsilon(g,s)\rangle_\epsilon = \int dx_1\, f(x_1)(U(t,s)gr(s))(x_1)$$
$$+ \int dx_1\, dx_2\, f(x_1)g(x_2)(U(t,0)\,U(s,0)h)(x_1,x_2)$$
$$+ \int_0^s ds' \int dx_1\, dx_2\,(U^*(t,s')f)(x_1)$$
$$\times (U^*(s,s')g)(x_2)(R(r(s'),r(s')))(x_1,x_2) \quad (3.10)$$

Remark. If the kernel of $U^*(t,s)$, in the distributional sense, is denoted by $U^*(t,s)(x_1,x_2), (U(t,s)f)(x_2) = \int dx_1\, U^*(t,s)(x_1,x_2)f(x_1)$, then the kernel of the covariance of the fluctuation field is

$$U^*(t,s)(x_1,x_2)r(x_2,s) + (U(t,0)\,U(s,0)h)(x_1,x_2)$$
$$+ \int_0^s ds' \left[U(t,s')\,U(s,s')R(r(s'),r(s'))\right](x_1,x_2) \qquad (3.11)$$

Remark. In thermal equilibrium $r(s) = zh_\beta$ and, since $R(h_\beta,h_\beta) = 0$, the third term of (3.11) vanishes. Let us define the linearized Boltzmann operator L by $Lf = L_{h_\beta}(h_\beta f)$. [$L_{r(t)}$ corresponds to linearizing as $r(t) + f$, whereas L corresponds to linearizing as $h_\beta(1+f)$.] Let $\mathcal{H} = L^2(\Lambda \times \mathbb{R}^3, zh_\beta(p)dq\,dp)$ with scalar product $\langle \cdot \mid \cdot \rangle$. L generates a contraction semigroup in $\mathcal{H}$. For the grand canonical ensemble $h = 0$ and the covariance is given by $\langle f | e^{L(t-s)} g\rangle$, which coincides with the result obtained in Ref. 1. For the canonical ensemble the constraint on the number of particles results in $h(x_1,x_2) = -zh_\beta(x_1)zh_\beta(x_2)$ and therefore the covariance is given by $\langle f|e^{L(t-s)}g\rangle - \langle f|P_1 g\rangle = \langle f - P_1 f | e^{L(t-s)}(g - P_1 g)\rangle$, where P_1 is the orthogonal projection onto the constant function.

Proof. (i) *the perturbation series.* Let $(V^\epsilon(t)r^\epsilon)_n = r_n^\epsilon(t)$ and let

$$(A(g)r^\epsilon)_n(x_1,\ldots,x_n) = \left[\sum_{j=1}^n g(x_j)\right] r_n^\epsilon(x_1,\ldots,x_n)$$
$$(A(g)r^\epsilon)_0 = 0$$
$$(B(g)r^\epsilon)_n(x_1,\ldots,x_n) = \int dx_{n+1}\, g(x_{n+1})r_{n+1}^\epsilon(x_1,\ldots,x_{n+1})$$

Then

$$\langle \xi^\epsilon(f,t)\xi^\epsilon(g,s)\rangle = \epsilon^{-2}\left[\langle X^\epsilon(f,t)X^\epsilon(g,s)\rangle_\epsilon - \langle X^\epsilon(f,t)\rangle_\epsilon \langle X^\epsilon(g,s)\rangle_\epsilon\right]$$
$$= \int dx_1\, f(x_1)\left\{V^\epsilon(t-s)\left[A(g) + \epsilon^{-2}B(g)\right]V^\epsilon(s)r^\epsilon\right\}_1(x_1)$$
$$- \epsilon^{-2}\int dx_1\, f(x_1)r_1^\epsilon(x_1,t)\int dx_2\, g(x_2)r_1^\epsilon(x_2,s) \qquad (3.12)$$

By Lanford's theorem

$$\lim_{\epsilon\to 0}\int dx_1\, f(x_1)(V^\epsilon(t-s)A(g)V^\epsilon(s)r^\epsilon)_1(x_1) = \int dx_1\, f(x_1)(V(t-s)A(g)V(s)r)_1(x_1) \tag{3.13}$$

Since $(V(s)r)_n = \prod r(s)$, the Boltzmann hierarchy (2.13) has to be solved with initial conditions

$$r_n(x_1,\ldots,x_n) = \left[\sum_{j=1}^{n} g(x_j)\right]\prod_{j=1}^{n} r(x_j,s) \tag{3.14}$$

One obtains

$$(V(t-s)r)_n(x_1,\ldots,x_n) = \sum_{j=1}^{n}\left\{(U(t,s)gr(s))(x_j)\prod_{\substack{k=1\\k\neq j}}^{n} r(x_k,s)\right\} \tag{3.15}$$

(3.15) together with (3.13) gives the first term of (3.10).

In order to write down the perturbation series for the $B(g)$ term, we have to define various collision operators. Let

$$(B(x)r^\epsilon)_n(x_1,\ldots,x_n) = r^\epsilon_{n+1}(x_1,\ldots,x_n,x) \tag{3.16}$$

$$(C^\epsilon(j,p_{n+1},\hat\omega_{n+1})r^\epsilon)_n(x_1,\ldots,x_n) = r^\epsilon_{n+1}(x_1,\ldots,x_n,q_j+\epsilon\hat\omega_{n+1},p_{n+1})\hat\omega_{n+1}\cdot(p_{n+1}-p_j) \tag{3.17}$$

for $j=1,\ldots,n$, and let

$$C^\epsilon_{j,n+1} = \int dp_{n+1}\,d\hat\omega_{n+1}\,C^\epsilon(j,p_{n+1},\hat\omega_{n+1}) \tag{3.18}$$

Then

$$\begin{aligned}
&\epsilon^{-2}\int dx_1\, f(x_1)(V^\epsilon(t-s)B(g)V^\epsilon(s)r^\epsilon)_1(x_1)\\
&\quad=\sum_{n=0}^{\infty}\int_{0\leqslant t_n\cdots\leqslant t_k\leqslant s\leqslant t_{k-1}\leqslant\cdots\leqslant t_1\leqslant t} dt_1\cdots dt_n\int dx_1\,dx_2\, f(x_1)g(x_2)\\
&\quad\times\sum_{j_1=1}\sum_{j_2=1,2}\cdots\sum_{j_{k-1}=1,3}^{k}\sum_{j_k=1}^{k+1}\\
&\quad\cdots\sum_{j_n=1}^{n+1}\big[S^\epsilon_1(t-t_1)C^\epsilon_{j_1,3}S^\epsilon_2(t_1-t_2)\\
&\quad\cdots S^\epsilon_k(t_{k-1}-s)B(x_2)\times S^\epsilon_{k+1}(s-t_k)C^\epsilon_{j_k,k+2}\\
&\quad\cdots C^\epsilon_{j_n,n+2}S^\epsilon_{n+2}(t_n)r^\epsilon_{n+2}\big](x_1)
\end{aligned} \tag{3.19}$$

The index k is determined by $t_k \leqslant s < t_{k-1}$.

We concentrate now on one particular term of the pertubation series

(3.19) and we will worry about convergence of the series uniformly in ϵ later on. Let us fix n, $0 < t_n < \cdots < t_k < s < t_{k-1} < \cdots < t_1 < t$ and a sequence $\{j_1, \ldots, j_n\}$. At time zero we have $n+2$ particles labeled $\{1, \ldots, n+2\}$. 1 is the label of the particle present at time t and 2 is the label of the particle adjoined at time s. $\{1, \ldots, n+2\}$ is partitioned into two sets M_1, M_2 depending on whether the particle labeled j_m is "mechanically connected" to either particle 1 or particle 2. Formally M_1 and M_2 are constructed by iteration: $1, j_1, \ldots, j_{k-1} \in M_1$, $2 \in M_2$, and $m+2 \in M_i$ if $j_m \in M_i$, $m = k, \ldots, n$. For simplicity particles with labels in M_i are called i particles.

The contribution to the pertubation series of this particular term is then

$$
\begin{aligned}
&I_+(\epsilon, t_1, \ldots, t_n, j_1, \ldots, j_n) \\
&\quad \equiv I_+(\epsilon) = \epsilon^{-2} \int_{\Delta_+(\epsilon)} dx_1\, dx_2\, dp_3\, d\hat{\omega}_3 \cdots dp_{n+2}\, d\hat{\omega}_{n+2}\, f(x_1) g(x_2) \\
&\qquad \times \big[S_1^\epsilon(t - t_1) C^\epsilon(j_1, p_3, \hat{\omega}_3) \cdots S_k^\epsilon(t_{k-1} - s) B(x_2) \\
&\qquad \times S_{k+1}^\epsilon(s - t_k) C^\epsilon(j_k, p_{k+2}, \hat{\omega}_{k+2}) \cdots S_{n+2}^\epsilon(t_n) r_{n+2}^\epsilon \big](x_1)
\end{aligned}
\tag{3.20}
$$

The definition of $\Delta_+(\epsilon) = \Delta_+(\epsilon, t_1, \ldots, t_n, j_1, \ldots, j_n)$ will be given in a moment.

We will use the following notational convention. The dependence on $t_1, \ldots, t_n, j_1, \ldots, j_n$ is suppressed unless necessary. Subsets of $\Delta_+(\epsilon)$ will be denoted by further arguments $\Delta_+(\epsilon, \ldots)$. The integrands are denoted by $I_+(\epsilon, \ldots)$ with arguments not yet summed over. For example, $I_+(\epsilon, t_1, \ldots, t_n, j_1, \ldots, j_n, x_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2})$ denotes the integrand of (3.20) and $I_+(\epsilon, t_1, \ldots, t_n)$ denotes $I_+(\epsilon, t_1, \ldots, t_n, j_1, \ldots, j_n)$ summed over $j_1, \ldots, j_n$. From the arguments appearing in I_+ it should be plain what quantity we consider. The limit of $I_+(\epsilon, \ldots)$ as $\epsilon \to 0$ is denoted by $I_+(\ldots)$.

The integrand of (3.20) has a simple constructive meaning. We start particle 1 at $x_1 = (q_1, p_1) = x_1(t)$. It evolves backwards in time under the hard sphere dynamics for a time span $t - t_1$ to $q_1(t_1), p_1(t_1)$. Then particle 3 is adjoined at $q_1(t_1) + \epsilon\hat{\omega}_3$ with momentum p_3. These two particles evolve backwards in time under the hard sphere dynamics for a time span $t_1 - t_2$, etc. At time s we adjoin the particle 2 at q_2 with momentum p_2. This construction defines, in principle, $x_1(0), \ldots, x_{n+2}(0)$ as a function of $x_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2}$ and therefore $r_{n+2}^\epsilon[x_1(0), \ldots, x_{n+2}(0)]$. The construction may break down because of two reasons. (i) The particle adjoined at any one of the time points $t_1, \ldots, s, \ldots, t_n$ overlaps with the particles already present at that time. Let $\tilde{\Delta}(\epsilon) \subset (\Lambda \times \mathbb{R}^3)^2 \times (\mathbb{R}^3 \times S^2)^n$ be the set of points for which such an overlap does not occur. (ii) At some point of the construction one reaches a singular configuration beyond which the hard sphere dynamics is not defined (cf. Section 2). In this case the dynamical trajectory to be constructed from $x_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2} \in \tilde{\Delta}(\epsilon)$ remains undefined. From the work of Alexander we know that the set of such points in $\tilde{\Delta}(\epsilon)$ forms a set of $dx_1\,dx_2\,dp_3\,d\hat{\omega}_3 \cdots dp_{n+2}\,d\hat{\omega}_{n+2}$ —measure zero. Excluding this set of measure zero from $\tilde{\Delta}(\epsilon)$ defines $\Delta_+(\epsilon)$.

We proceed by writing out the pertubation series for $\int dx_1\,dx_2\, f(x_1) g(x_2) r_1^\epsilon(x_1,t) r_1^\epsilon(x_2,s)$. We note that this series is given by (3.19) together with the rule that collisions between 1-particles and 2-particles are ignored. To be more precise: Fix again $t_1, \ldots, t_n$ and $j_1, \ldots, j_n$. Then M_1 and M_2 are uniquely defined. Let $\bar{S}_m^\epsilon(t)$ denote the time evolution of m hard spheres, such that collisions between particles with labels in $M_1 \cap \{1, \ldots, m\}$ and particles with labels in $M_2 \cap \{1, \ldots, m\}$ are ignored. Let us define

$$\begin{aligned} I_-(\epsilon, t_1, \ldots, t_n, j_1, \ldots, j_n) &\\ \equiv I_-(\epsilon) = \epsilon^{-2} \int_{\Delta_-(\epsilon)} & dx_1\,dx_2\,dp_3\,d\hat{\omega}_3 \cdots dp_{n+2}\,d\hat{\omega}_{n+2}\, f(x_1) g(x_2) \\ & \times \left[\bar{S}_1^\epsilon(t-t_1) C^\epsilon(j_1, p_3, \hat{\omega}_3) \cdots \bar{S}_{n+2}^\epsilon(t_n) \bar{r}_{n+1}^\epsilon \right](x_1) \quad (3.21) \end{aligned}$$

[cf. (3.20), where $\bar{r}_{n+2}^\epsilon(x_1, \ldots, x_{n+2}) = r_m^\epsilon(x_{k_1}, \ldots, x_{k_m}) r_1^\epsilon(x_{k_{m+1}}, \ldots, x_{k_{m+1}})$, $\{k_1, \ldots, k_m\} = M_1, \{k_{m+1}, \ldots, k_{m+1}\} = M_2$, and where $\Delta_-(\epsilon)$ is defined as $\Delta_+(\epsilon)$ ignoring collisions between 1-particles and 2-particles.] Then

$$\begin{aligned} \int & dx_1\,dx_2\, f(x_1) g(x_2) r_1^\epsilon(x_1,t) r_1^\epsilon(x_2,s) \\ &= \sum_{n=0}^{\infty} \int_{0 \leqslant t_n \cdots \leqslant t_k \leqslant s \leqslant t_{k-1} \leqslant \cdots \leqslant t_1 \leqslant t} dt_1 \ldots dt_n \\ &\quad \times \sum_{j_1=1} \sum_{j_2=1,3} \cdots \sum_{j_n=1}^{n+1} I_-(\epsilon, t_1, \ldots, t_n, j_1, \ldots, j_n) \quad (3.22) \end{aligned}$$

[cf. (3.19)].

(*ii*) *Partition of* $\Delta_+(\epsilon)$ *and* $\Delta_-(\epsilon)$. For $x_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2} \in \Delta_+(\epsilon)$, let $\tau = \tau(x_1, x_2, p_3, \ldots, \hat{\omega}_{n+2})$ be the time of the last collision between 1-particles and 2-particles. We drop a set of measure zero from $\Delta_+(\epsilon)$ corresponding to a simultaneous collision of two pairs of 1-particles and 2-particles at time τ. After time τ only collisions between 1-particles by themselves and between 2-particles by themselves occur. Then we define

$$\begin{aligned} \Delta_+(\epsilon, n+1) = \{ x_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2} \in \Delta_+(\epsilon) \mid & \\ \text{no collisions between 1-particles} & \\ \text{and 2-particles occur} \} & \quad (3.23) \end{aligned}$$

and

$$\Delta_+(\epsilon,m) = \{x_1, x_2, p_3, \ldots, \hat{\omega}_{n+2} \in \Delta_+(\epsilon) \mid \tau \in [t_{m+1}, t_m)\} \quad (3.24)$$

with $m = k-1, \ldots, n$, $t_{k-1} = s$, $t_{n+1} = 0$. The decomposition $\Delta_-(\epsilon)$ is defined in the same way. τ is now the time of the last overlap between 1-particles and 2-particles. The partition (3.23) of $\Delta_+(\epsilon)$, respectively, (3.24) of $\Delta_-(\epsilon)$, is inserted in (3.20), respectively, in (3.21).

(*iii*) *Discussion of* $I_+(\epsilon, n+1)$ *and* $I_-(\epsilon, n+1)$. Clearly $\Delta_+(\epsilon, n+1) = \Delta_-(\epsilon, n+1)$ and on $\Delta_+(\epsilon, n+1)$ (3.20) and (3.21) are identical except for the initial condition. Subtracting both terms the initial correlation functions are $\epsilon^{-2}(r_{n+2}^{\epsilon} - \bar{r}_{n+2}^{\epsilon})$. By (F1) and (F2) they satisfy the conditions (C1) and (C2) of Lanford's theorem. So

$$\begin{aligned}
&\lim_{\epsilon\to 0} I_+(\epsilon, n+1) - I_-(\epsilon, n+1) \\
&\quad = \int dx_1\, dx_2\, f(x_1) g(x_2) \big[S_1(t-t_1) C_{j_1,3} \cdots S_k(t_{k-1}-s) \\
&\quad\quad \times B(x_2) S_{k+1}(s-t_k) \cdots C_{j_n,n+2} S_{n+2}(t_n) r_{n+2} \big](x_1) \qquad (3.25)
\end{aligned}$$

with

$$r_{n+2}(x_1, \ldots, x_{n+2}) = \sum_{i\in M_1} \sum_{j\in M_2} \left\{ h(x_i, x_{m+j}) \prod_{\substack{k=1 \\ k\neq i,\, k\neq m+j}}^{n+2} r(x_k) \right\} \quad (3.26)$$

Note that (3.26) is symmetric only under permutations of particles within one group.

We add up all terms of the pertubation series of the form (3.25). This yields

$$\int dx_1\, dy_1\, f(x_1) g(y_1) \big(V(t-s) \big[W(s) r \big]_{\cdot,1}(y_1) \big)_1 (x_1) \qquad (3.27)$$

$W(t)$ is the solution operator to the two-particle Boltzmann hierarchy

$$\begin{aligned}
&\frac{\partial}{\partial t} r_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, t) \\
&\quad = \left(-\sum_{i=1}^{m} p_i \frac{\partial}{\partial q_i} - \sum_{j=1}^{n} p_j \frac{\partial}{\partial q_j} \right) r_{m,n}(x_1, \ldots, y_n, t) \\
&\quad\quad + \sum_{i=1}^{m} (C_{i,m+1} r_{m+1,n})(x_1, \ldots, y_n, t) \\
&\quad\quad + \sum_{j=1}^{n} (C_{j,n+1} r_{m,n+1})(x_1, \ldots, y_n, t) \qquad (3.28)
\end{aligned}$$

The initial conditions r in (3.27) are

$$r_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_n)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{h(x_i,y_j)\prod_{\substack{k=1\\k\neq i}}^{m}\prod_{\substack{k'=1\\k'\neq j}}^{n}r(x_k)r(y_{k'})\right\} \tag{3.29}$$

For $0\leqslant t\leqslant t_0$ the solution is obtained as

$$\begin{aligned}&r_{m,n}(x_1,\ldots,x_m,y_1,\ldots,y_n,s)\\&\quad=\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{(U(s,0)U(s,0)h)(x_i,y_j)\prod_{\substack{k=1\\k\neq i}}^{m}\prod_{\substack{k'=1\\k'\neq j}}^{n}r(x_k,s)r(y_{k'},s)\right\}\end{aligned} \tag{3.30}$$

The components $\{r_{m+1}(x_1,\ldots,x_m,y_1,s)\,|\,m=1,2,\ldots\}$ define the initial conditions for $V(t-s)$ yielding

$$\begin{aligned}&\left[V(t-s)r(y_1,s)\right]_m(x_1,\ldots,x_m)\\&\quad=\sum_{i=1}^{m}\left\{(U(t,0)U(s,0)h)(x_i,y_1)\prod_{\substack{k=1\\k\neq i}}^{m}r(x_k,t)\right\}\end{aligned} \tag{3.31}$$

The first component of (3.31) results then in the second term of (3.10).

(*iv*) *Discussion of $I_+(\epsilon,m)$, $m=k-1,\ldots,n$.* In (iii) the prefactor ϵ^{-2} was cancelled by clustering of the initial state. For $I_+(\epsilon,m)$, $m\leqslant n$, we will have to cancel the prefactor ϵ^{-2} by forcing one additional collision.

We partition $\Delta_+(\epsilon,m)$ into $\{\Delta_+(\epsilon,m,i,j)\}$, where

$$\begin{aligned}\Delta_+(\epsilon,m,i,j)=\{x_1,x_2,p_3,\hat\omega_3,\ldots,p_{n+2},\hat\omega_{n+2}\in\Delta_+(m,\epsilon)\,|\,&\text{the collision}\\&\text{at time }\tau\in[t_{m+1},t_m)\text{ is between}\\&\text{particle }i\in M_1\text{ and particle }j\in M_2\}\end{aligned}$$

Let us define

$$\begin{aligned}&(R^\epsilon(\hat\omega,i,j)r^\epsilon)_n(x_1,\ldots,x_n)\\&\quad=\hat\omega\cdot(p_j-p_i)r_n^\epsilon(x_1,\ldots,q_i,p_i,\ldots,q_i+\epsilon\hat\omega,p_j,\ldots,x_n)\end{aligned} \tag{3.32}$$

Then we have the identity

$$\begin{aligned}I_+(\epsilon,m,i,j)=\int_{\tilde\Delta_+(\epsilon,m,i,j)}&d\tau\,d\hat\omega\,dp_1\,dx_2\,dp_3\,d\hat\omega_3\cdots dp_{n+2}\,d\hat\omega_{n+2}\\&\times f(q_1,p_1)g(x_2)\big[S_1^\epsilon(t-t_1)C^\epsilon(j_1,p_3,\hat\omega_3)\\&\cdots S_k^\epsilon(t_{k-1}-s)B(x_2)S_{k+1}^\epsilon(s-t_k)\\&\cdots S_{m+2}^\epsilon(t_m-\tau)R^\epsilon(\hat\omega,i,j)S_{m+2}^\epsilon(\tau-t_{m+1})\\&\times C^\epsilon(j_{m+1},p_{m+3},\hat\omega_{m+3})\cdots S_{n+2}^\epsilon(t_n)r_{n+2}^\epsilon\big](q_1,p_1)\end{aligned} \tag{3.33}$$

(3.33) results from a change of variables of q_1 to τ, $\hat\omega$ achieved in the following way: For $q_1,p_1,x_2,p_3,\hat\omega_3,\ldots,p_{n+2},\hat\omega_{n+2}\in\Delta_+(\epsilon,m,i,j)\,|\,\tau$

$\in (t_{m+1}, t_m)$ is the time of a last collision between particle i and particle j such that after τ there are no further collisions between 1-particles and 2-particles and $\hat{\omega}$ is given by $q_j(\tau) = q_i(\tau) + \epsilon\hat{\omega}$. This defines $\hat{\omega}$ and τ as functions on $\Delta_+(\epsilon, m, i, j)$. Let $\tilde{\Delta}_+(\epsilon, m, i, j)$ be the image of $\Delta_+(\epsilon, m, i, j)$ under this mapping. Conversely, since 1-particles and 2-particles do not interact after time τ, the history of the 2-particles between time τ and s and therefore $q_j(\tau)$ is well defined and independent of the coordinates of the 1-particles. Let us construct the history of the 1-particles between time τ and t. Because of the periodic boundary conditions, if q_1 is shifted to $q_1 + a$ while keeping $p_1, p_3, \hat{\omega}_3, \ldots$ fixed, then $q_i(\tau)$ is shifted to $q_i(\tau) + a$. Therefore q_1 is uniquely defined by requiring

$$q_j(\tau) = q_i(\tau) + \epsilon\hat{\omega}, \qquad (p_j(\tau) - p_i(\tau)) \cdot \hat{\omega} > 0 \tag{3.34}$$

By construction $q_1 \to \tau, \hat{\omega}$ is one-to-one as a map from $\Delta_+(\epsilon, i, j, m)$ to $\tilde{\Delta}_+(\epsilon, i, j, m)$.

The Wronskian $\partial(q_1(\tau))/\partial(\tau\hat{\omega}) = \epsilon^2\hat{\omega} \cdot (p_j(\tau) - p_i(\tau))$ and, because of periodic boundary conditions, the Wronskian $\partial(q_1)/\partial(q_i(\tau)) = 1$. Therefore the volume element transforms as

$$dq_1 = \epsilon^2\hat{\omega} \cdot (p_j(\tau) - p_i(\tau))\, d\tau\, d\hat{\omega} \tag{3.35}$$

which is taken into account through the definition of $R^\epsilon(\hat{\omega}, i, j)$.

(*iv*) *Uniform Bound.* The limit $\epsilon \to 0$ is proved by dominated convergence. So first we should obtain a bound uniform in ϵ. In (3.33)

$$\begin{aligned} &|I_+(\epsilon, t, \hat{\omega}, p_1, x_2, \ldots, p_{n+2}, \hat{\omega}_{n+2}, j_1, \ldots, j_n, i, j)| \\ &\quad \leqslant (|p_{j_1}(t_1)| + |p_3|) \cdots (|p_i(\tau)| + |p_j(\tau)|) \cdots (|p_{j_n}(t_n)| + |p_{n+2}|) \\ &\quad \times M z^{n+2} \chi^\epsilon_{n+2}(q_1(0), \ldots, q_{n+2}(0)) \prod_{j=1}^{n+2} h_\beta(p_j(0)) \end{aligned} \tag{3.36}$$

since $|\hat{\omega} \cdot (p_i - p_j)| \leqslant |p_i| + |p_j|$ and by the assumed bound (C2) for r^ϵ_{n+2}. Here $\chi^\epsilon_n(q_1, \ldots, q_n) = 0$ whenever $|q_i - q_j| < \epsilon$ and $\chi^\epsilon_n(q_1, \ldots, q_n) = 1$ otherwise. In (3.36) we sum first over all pairs i, j, and then over all $j_1, \ldots, j_n$. Then, using

$$\sum_{j=1}^{n} |p_j| \leqslant \left(n \sum_{j=1}^{n} p_j^2 \right)^{1/2} \tag{3.37}$$

we have

$$\begin{aligned} |I_+(\epsilon, \ldots, \hat{\omega}_{n+2})| &\leqslant (|p_1(t_1)| + |p_3|) \cdots \left((m+2) \sum_{j=1}^{m+2} p_j(\tau)^2 \right)^{1/2} \\ &\quad \times (m+2) \cdots \left\{ \left[(n+1) \sum_{j_n=1}^{n+1} p_{j_n}(t_n)^2 \right]^{1/2} + (n+1)|p_{n+2}| \right\} \\ &\quad \times M z^{n+2} \chi^\epsilon_{n+2}(q_1(0), \ldots, q_{n+2}(0)) \prod_{j=1}^{n+2} h_\beta(p_j(0)) \end{aligned} \tag{3.38}$$

By stationarity of the equilibrium distribution

$$\chi_{n+2}^{\epsilon}(q_1(0), \ldots, q_{n+2}(0)) \prod_{j=1}^{n+2} h_\beta(p_j(0))$$
$$= \chi_{n+2}^{\epsilon}(q_1(t_n), \ldots, q_{n+2}(t_n)) \prod_{j=1}^{n+2} h_\beta(p_j(t_n))$$
$$\leqslant \chi_{n+1}^{\epsilon}(q_1(t_n), \ldots, q_{n+1}(t_n)) \prod_{j=1}^{n+1} h_\beta(p_j(t_n)) h_\beta(p_{n+2}) \tag{3.39}$$

Together with conservation of energy this implies the bound

$$|I_+(\epsilon, \ldots, \hat{\omega}_{n+2})| \leqslant (|p_1(t_1)| + |p_3|) \cdots \left[(n+1) \sum_{j_n=1}^{n+1} p_{j_n}(t_{n-1})^2\right]^{1/2}$$
$$+ ((n+1)|p_{n+2}|) M z^{n+2} \chi_{n+1}^{\epsilon}(q_1(t_{n-1}), \ldots, q_{n+1}(t_{n-1}))$$
$$\times \prod_{j=1}^{n+1} h_\beta(p_{n+1}(t_{n-1})) h_\beta(p_{n+2}) \tag{3.40}$$

Finally, by iteration,

$$|I_+(\epsilon, \ldots, \hat{\omega}_{n+2})| \leqslant (|p_1| + |p_3|) \cdots (m+2)\left[(m+2) \sum_{j=1}^{m+2} p_j^2\right]^{1/2}$$
$$\cdots \left\{\left[(n+1) \sum_{j_n=1}^{n+1} p_{j_n}^2\right]^{1/2} + (n+1)|p_{n+2}|\right\}$$
$$\times M z^{n+2} \prod_{j=1}^{n+2} h_\beta(p_j) \tag{3.41}$$

We add the contributions of $\Delta(\epsilon, m)$, $m = k-1, \ldots, n$, and perform the multiple time integrations. This yields then a bound on the pertubation series for particle histories with at least one collision between 1-particles and 2-particles:

$$\sum_{k=1}^{\infty} \sum_{m=k-1}^{n} |I_+(\epsilon, m)| \leqslant c \sum_{n=0}^{\infty} \frac{t^n}{n!} (4\pi z)^n \int dp_1 \cdots dp_{n+2} \prod_{j=1}^{n+2} h_\beta(p_j)$$
$$\times \left[\left(2 \sum_{j_1=1}^{2} p_{j_1}^2\right)^{1/2} + 2|p_3|\right]$$
$$\cdots \left[\left((n+1) \sum_{j_n=1}^{n+1} p_{j_n}^2\right)^{1/2} + (n+1)|p_{n+2}|\right]$$
$$\times \left\{\sum_{m=0}^{n} (m+2)^{3/2} \left(\sum_{j=1}^{m+2} p_j^2\right)^{1/2}\right\} \tag{3.42}$$

(3.42) differs from the bound in Lanford's theorem only by the term in the curly bracket. Its contribution is of the order $n^{5/2}$ to the nth term of the sum. So (3.42) converges provided $t \leqslant t_0(z, \beta)$.

(ivb) Term-by-Term Convergence. We want to prove the limit as $\epsilon \to 0$ of $I_+(\epsilon, m, i, j)$ [cf. (3.33)]. By (3.41) dominated convergence may be applied. Let us construct the particle history as described before with particles now considered as noninteracting point particles. More precisely: We place a point particle at $q_2 = q_2(s)$ with momentum $p_2 = p_2(s)$. It evolves freely backwards in time for a time span $s - t_k$. Let $j_1 \in M_2$ be the smallest element in M_2 with $j_1 > 2$. Then we adjoin the point particle j_1 at $q_{j_1} = q_2(t_k)$ with momentum p_{j_1}. To the pair $p_2(t_k), p_{j_1}$ we associate the incoming momenta $p_2'(t_k), p_{j_1}'$ according to $\hat{\omega}_{j_1} \cdot (p_2(t_k) = p_2'(t_k)$ and $p_{j_1} = p_{j_1}'$, if $(p_2(t_k) - p_{j_1}) \cdot \hat{\omega}_{j_1} > 0)$. Both particles evolve then freely a time span $t_k - t_{k+1}$ backwards in time, etc., up to time τ. Then the history of the 1-particles is constructed in the same way, where q_1 is uniquely determined through the condition $q_i(\tau) = q_j(\tau)$. We require also that $(p_j(\tau) - p_i(\tau)) \cdot \hat{\omega} > 0$ which is an implicit condition on range of $p_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{m+2}, \hat{\omega}_{m+2}$. By (2.1) $\hat{\omega}$ determines the incoming momenta $p_i'(\tau)$ and $p_j'(\tau)$. Then, finally, the particle history is completed up to time zero. $\Delta_+(m, i, j)$ denotes the allowed set of $\tau, \hat{\omega}, p_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2}$. Let $\Delta_+(0, m, i, j) \subset \Delta_+(m, i, j)$ be the set of points such that its corresponding particle history has, besides the ones described above, no further points where two particles spatially coincide at some time.

Let

$$(R(0, \hat{\omega}, i, j)r)_n(x_1, \ldots, x_n) = \hat{\omega} \cdot (p_j - p_i) \times r_n(x_1, \ldots, q_i, p_i', \ldots, q_i, p_j', \ldots, x_n) \tag{3.43}$$

and let

$$\begin{aligned}(C(j, p_{n+1}, \hat{\omega}_{n+1})r)_n(x_1, \ldots, x_n) &= \hat{\omega} \cdot (p_{n+1} - p_j)(r_{n+1}(x_1, \ldots, q_j, p_j', \ldots, q_j, p_{n+1}') \\ &\quad - r_{n+1}(x_1, \ldots, q_j, p_j, \ldots, q_j, p_{n+1}))\end{aligned} \tag{3.44}$$

Let $A \subset \Delta_+(0, m, i, j)$ be compact. Then ϵ can be chosen so small that $A \subset \tilde{\Delta}_+(\epsilon, m, i, j)$. Let $x_j(\epsilon, 0)$ be the position and momentum of the jth particle at time zero considered as a function of $\tau, \hat{\omega}, p_1, x_2, p_3, \hat{\omega}_3, \ldots, p_{n+2}, \hat{\omega}_{n+2}$ as constructed in part (iv) of the proof and let $x_j(0)$ be the position and momentum of the jth particle at time zero as constructed above. Then on A, $x_1(\epsilon, 0), \ldots, x_{n+2}(\epsilon, 0)$ are continuous, vary in a compact set of $\Gamma_{n+2}(0)$, and converge as $\epsilon \to 0$ to $x_1(0), \ldots, x_{n+2}(0)$. (C2) implies then that the integrand of (3.33) converges on A. Since $\Delta_+(m,i,j) \backslash \Delta_+(0,m,i,j)$ is of measure zero, by dominated convergence

$$\lim_{\epsilon\to 0} I_+(\epsilon,m,i,j) = \int_{\Delta_+(m,i,j)} d\tau\, d\hat\omega\, dp_1\, dx_2 \times dp_3\, d\hat\omega_3 \cdots dp_{n+2}\, d\hat\omega_{n+2}$$
$$\times f(q_1,p_1)\,g(x_2)\big[S_1(t-t_1)C(j_1,p_3,\hat\omega_3)$$
$$\cdots S_k(t_{k-1}-s)B(x_2)S_{k+1}(s-t_k)$$
$$\cdots S_{m+2}(t_m-\tau)R_+(0,\hat\omega,i,j)$$
$$\times S_{m+2}(\tau-t_{m+1})C(j_{m+1},p_{m+3},\hat\omega_{m+3})$$
$$\cdots S_{n+2}(t_n)r_{n+2}\big](q_1,p_1) \tag{3.45}$$

with $r_{n+2}(x_1,\ldots,x_{n+2}) = \prod_{j=1}^{n+2} r(x_j)$.

Let

$$(R_+(\hat\omega,i,j)r_n)(x_1,\ldots,x_n)$$
$$= \hat\omega\cdot(p_j-p_i)\times r_n(x_1,\ldots,q_i,p_i',\ldots,q_j,p_j',\ldots,x_n)\delta(q_i-q_j) \tag{3.46}$$

We reintroduce the q_1 integration and obtain

$$\lim_{\epsilon\to 0} I_+(\epsilon,m,i,j)$$
$$= \int_{t_{m+1}(\Lambda\times\mathbb{R}^3)^2\times(\mathbb{R}^3\times S^2)^n}^{t_m} d\tau \int dx_1\, dx_2\, dp_3\, d\hat\omega_3 \cdots dp_{n+2}\, d\hat\omega_{n+2}$$
$$\times \int_{\hat\omega\cdot(p_j-p_i)\geqslant 0} d\hat\omega\, f(x_1)g(x_2)\big[S_1(t-t_1)C(j_1,p_3,\hat\omega_3)$$
$$\cdots S_k(t_{k-1}-s)\times B(x_2)S_{k+1}(s-t_k)$$
$$\cdots S_{m+2}(t_m-\tau)R_+(\hat\omega,i,j)S_{m+2}(\tau-t_{m+1})$$
$$\cdots S_{n+2}(t_n)r_{n+2}\big](x_1) \tag{3.47}$$

Remark. If specularly reflecting boundary conditions are imposed, then the function $q_1\to\tau,\hat\omega$ is not one-to-one, in general. In fact, because of focusing, there may be infinitely many q_1's satisfying (3.34). In principle, one should partition $\Delta_+(\epsilon,i,j,m)$ such that on each element of the $q_1\to\tau$, $\hat\omega$ is one-to-one. However, even then the difficulty of obtaining uniform bounds on the Wronskian remains. Formally (3.47) is still correct. Instead of (3.35) the volume element transforms as

$$dq_1 = \epsilon^2(\partial(q_1)/\partial(q_i(\tau)))\hat\omega\cdot(p_j(\tau)-p_i(\tau))\,d\tau\, d\hat\omega \tag{3.48}$$

In the limit $\epsilon \to 0$, $\partial(q_1)/\partial(q_i(\tau))$ is the correct volume element for the δ function appearing in (3.47).

(v) *Discussion of* $I_-(\epsilon, m)$, $m = k-1, \ldots, n$. The proof of convergence of the term $I_-(\epsilon, m)$ parallels the one for $I_+(\epsilon, m)$ given in part (iv). We partition $\Delta_-(\epsilon, m)$ into $\Delta_-(\epsilon, m, i, j)$ according to the pair of particles i and j touching at time τ. Then the q_1 integration is changed to the $d\tau\, d\hat{\omega}$ integration. q_1 is still defined by (3.34). Particles i and j touch at time τ. Backwards in time they simply pass through each other. So $p_i(\tau)$ and $p_j(\tau)$ are the same just before and just after touching. The uniform bound for $I_-(\epsilon, \ldots, \hat{\omega}_{n+2})$ is identical to (3.41).

Let

$$(R_-(\hat{\omega}, i, j)r_n)(x_1, \ldots, x_n)$$
$$= \hat{\omega} \cdot (p_j - p_i) r_n(x_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, x_n)\delta(q_i - q_j) \quad (3.49)$$

for $\hat{\omega} \cdot (p_j - p_i) \geqslant 0$. Then, by the same argument as used before, $I_-(\epsilon, m, i, j)$ converges to (3.47) with $R_+(\hat{\omega}, i, j)$ replaced by $R_-(\hat{\omega}, i, j)$ as $\epsilon \to 0$.

Adding $I_+(m, i, j)$ and $-I_-(m, i, j)$ results in (3.47) with $R_+(\hat{\omega}, i, j)$ replaced by $R(\hat{\omega}, i, j) \equiv R_+(\hat{\omega}, i, j) - R_-(\hat{\omega}, i, j)$.

Finally we have to sum the pertubation series. One obtains

$$\int_0^s d\tau \int dx_1\, dy_1\, f(x_1) g(y_1) \big(V(t-s)\big[W(s-\tau) r(\tau)\big]_{\cdot,1}(y_1)\big)_1(x_1) \quad (3.50)$$

[cf. (3.27)], with initial condition

$$r_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, \tau)$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ R(r(\tau), r(\tau))(x_i, y_j) \prod_{\substack{k=1 \\ k \neq i}}^{m} \prod_{\substack{k'=1 \\ k' \neq j}}^{n} r(x_k, \tau) r(y_{k'}, \tau) \right\} \quad (3.51)$$

By the same argument as in part (iii), (3.50) with initial condition (3.51) results in the last term of (3.10). ■

4. CONCLUSIONS

Let us assume that the limiting random field $\xi(f, t)$ is Gaussian. Then $\xi(f, t)$ has mean zero and covariance (3.10). In formal derivations, usually, one tries to write down a stochastic partial differential equation for the time evolution of the fluctuation field. Let us define the fluctuation field pointwise by $\xi(f, t) = \int dq\, dp\, f(q, p) \xi(q, p, t)$. Then the fluctuations around the solution of the Boltzmann equation are governed by

$$\frac{\partial}{\partial t} \xi(q, p, t) = -p \frac{\partial}{\partial q} \xi(q, p, t) + C_{r(t)} \xi(q, p, t) + F(q, p, t) \quad (4.1)$$

[If $\xi(q, p, 0)$ is Gaussian with mean zero and covariance (3.3), then the solution of (4.1) with this initial condition is Gaussian with mean zero and covariance (3.10).] $C_{r(t)}$ is the linearized Boltzmann collision operator defined in (3.7). $F(q, p, t)$ is a Gaussian random force with mean zero and covariance

$$\begin{aligned}\int dp\, d\bar{p}\; g(p) f(\bar{p}) \left\langle F(q,p,t) F(\bar{q},\bar{p},\bar{t}) \right\rangle \\ = \tfrac{1}{2}\delta(t-\bar{t})\delta(q-\bar{q}) \int_{\hat{\omega}\cdot(p_1-p)\geqslant 0} dp\, dp_1\, d\hat{\omega}\, \hat{\omega}\cdot(p_1-p) \\ \times r(q,p_1',t) r(q,p',t) \{ g(p_1') + g(p') - g(p_1) - g(p) \} \\ \times \{ f(p_1') + f(p') - f(p_1) - f(p) \}\end{aligned} \tag{4.2}$$

The evolution equation (4.1) contains linearized field equations which evolve deterministically the fluctuations present in the initial state. (In the case of the Vlasov equation that is all what enters, i.e., $F \equiv 0$ for fluctuations in the mean field limit.) In addition (4.1) contains a fluctuating force due to the microscopic background. Our proof singles out precisely those microscopic events which produce this random force and shows that all other dynamical events are of probability zero in the low-density limit.

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